ELSEVIER

Journal of Geometry and Physics 36 (2000) 163–177

JOURNAL OF GEOMETRY AND PHYSICS

Intersection numbers on moduli spaces and symmetries of a Verlinde formula II

R. Herrera

Department of Mathematics, Yale University, 10 Hillhouse Avenue, PO Box 208283, New Haven, CT 06520-8283, USA

Received 23 February 1999; received in revised form 7 March 2000

Abstract

We investigate the geometry and topology of a moduli space of orthogonal vector bundles on a hyperelliptic Riemann surface, and derive results on intersection pairings by means of twistor transform and index calculations. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 81T30; 83E30

Sub. Class.: Strings

Keywords: Hyperelliptic Riemann surfaces; Moduli spaces; Verlinde formula

1. Introduction

In this note, we study certain moduli spaces $\mathcal{M}_{g,n}$ of rank $2n$ orthogonal vector bundles over hyperelliptic Riemann surfaces Σ of genus g, generalising some of the results proved in [6,7,15]. Some members of this family of moduli spaces are very familiar; for example, $\mathcal{M}_{g,1}$ is the Jacobian of Σ, and $\mathcal{M}_{g,2}$ is the moduli space of rank 2 holomorphic stable vector bundles over Σ with fixed and odd determinant (cf. [12]). Ramanan [12] identified $\mathcal{M}_{g,n}$ with a complete intersection of two 'quadric varieties' in a complex Grassmannian, which shows the existence of a positive line bundle L over $\mathcal{M}_{g,n}$. Hence, it is natural to study the Hilbert polynomial $\dim(H^0(\mathcal{M}_{g,n}, \mathcal{O}(L^k)))$ as a direct generalisation of the Verlinde formula of $\mathcal{M}_{g,2}$ (cf. [3,16,17,19]), as well as the intersection theory of $\mathcal{M}_{g,n}$ (cf. [2,8,17,19]). We derive our results as in [7], by using the embedding of $\mathcal{M}_{g,n}$ in a homogeneous space.

In Section 2, we give the definition of $\mathcal{M}_{g,n}$ and find a K-theoretic decomposition of its holomorphic tangent bundle. In Section 3, we prove some vanishings and symmetries of

E-mail address: herrera@math.yale.edu (R. Herrera).

PII: S0393-0440(00)00021-8

holomorphic Euler characteristics for virtual vector bundles and compute the *Verlinde-type formula* $\dim(H^0(\mathcal{M}_{g,n}, \mathcal{O}(L^k)))$ as a determinant of a matrix whose entries are Verlinde dimensions themselves. From this formula, we determine the symplectic volume of $\mathcal{M}_{g,n}$. In Section 4, we generalise results from [6,7], by computing all the intersection numbers of a subring of $H^*(\mathcal{M}_{g,g-1})$ generated by two quaternionic-related classes. Furthermore, we prove generalisations of the Newstead conjectures for $\mathcal{M}_{g,g-1}$ (cf. [11, Conjecture (a)–(c)]).

2. Moduli spaces

Let Σ_g be a hyperelliptic Riemann surface of genus g with involution $\iota : \Sigma \to \Sigma$ and Weierstrass points $\{\omega_1, \ldots, \omega_{2g+2}\}$. Consider the special Clifford group $SC(2n) = \mathbb{C}^* \times_{\mathbb{Z}_2} Spin(2n)$, which fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{C}^* & \to & SC(2n) & \to & SO(2n) & \to & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \to & \mathbb{Z}_2 & \to & Spin(2n) & \to & SO(2n) & \to & 1 \end{array}.$$

Let $\mathcal{M}_{g,n}$ denote the moduli space of semistable, holomorphic, rank $2n$, vector bundles E over Σ_g with the following properties:

- E is an (orthogonal) vector bundle with structure group $SO(2n)$ and with a lift of structure group to $SC(2n)$.
- E is ι-invariant, i.e. there is a lift of ι to E (denoted by the same symbol) such that $E \cong \iota^* E$. Thus, we have the restrictions of ι to the fibres over the Weierstrass points $\iota : E_{\omega_j} \to E_{\omega_j}$ for all $j = 1, \ldots, 2g+2$. Since $\iota^2 = 1$, the eigenvalues of ι on these fibres are ± 1, and we denote the eigenspace by $E^{\pm}_{\omega_j}$.
- E is such that $\dim((E \otimes \Lambda)^-_{\omega_j}) = 1$ for all $j = 1, \ldots, 2g+2$, where Λ is an ι-invariant line bundle over Σ of degree $2g-1$.

Example.

1. *Case* $n = 1$. Since $SO(2) \cong U(1)$, $\mathcal{M}_{g,1}$ is the Jacobian $J(\Sigma)$ of Σ_2 (cf. [12]).
2. *Case* $n = 2$. The special Clifford group is

$$SC(4) = \{(A, B) \in Gl(2) \times Gl(2) \mid \det(A) \cdot \det(B) = 1\}$$

   and the homomorphism $SC(4) \to SO(4)$ is given by $(A, B) \to A \otimes B$. Thus, a $SC(4)$-bundle is essentially a pair of $Gl(2)$-bundles M, N with $\det(M) \otimes \det(N) = 1$, a trivial bundle. Since the Clifford group $C(4)$ does not distinguish between M and N, we have that $\mathcal{M}_{g,2}$ is the moduli space of (stable) vector bundles of rank 2 and fixed odd determinant (cf. [12]).

Ramanan proved in [12, Theorem 3] that $\mathcal{M}_{g,n}$ is isomorphic to the variety of $(g+1-n)$-dimensional subspaces of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the two quadratic forms:

$$\sum_{i=1}^{2g+2} y_i^2, \qquad \sum_{i=1}^{2g+2} \omega_i y_i^2. \tag{1}$$

Therefore, we have a holomorphic embedding of $\mathcal{M}_{g,n}$ into the complex partial flag manifold

$$\mathcal{F}_{g,n} = \frac{SO(2g+2)}{U(g+1-n) \times SO(2n)},$$

which clearly parametrises the $(g+1-n)$-dimensional subspaces of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the first quadratic form. The flag manifold $\mathcal{F}_{g,n}$ is a twistor space for

$$\mathcal{G}_{g,n} = \frac{SO(2g+2)}{SO(2g+2-2n) \times SO(2n)},$$

since the fibre $SO(2g+2-2n)/U(g+1-n)$ parametrises orthogonal, almost-complex structures on the real oriented $(2g+2-2n)$-dimensional subspaces of $\mathbb{R}^{2g+2}$, which are compatible with the orientation (cf. [1,14]).

Let Q, W denote the duals of the tautological complex vector bundles over $\mathcal{F}_{g,n}$ with fibres $\mathbb{C}^{g+1-n}$, $\mathbb{C}^{2n}$ and structure groups $U(g+1-n)$, $SO(2n)$, respectively. The second quadratic form determines a holomorphic, non-degenerate section of the second symmetric tensor power S^2Q of Q, whose zero-set is precisely $\mathcal{M}_{g,n}$. Thus, we know that $\mathcal{M}_{g,n}$ is a smooth complex manifold of complex dimension $(2n-1)(g+1-n)$.

The splitting of the standard representation of $SO(2g+2)$ on $\mathbb{C}^{2g+2}$ under $U(g+1-n) \times SO(2n)$ yields

$$Q^* \oplus Q \oplus W = 2g+2. \tag{2}$$

This implies that

$$\mathfrak{so}(2g+2)_c \cong (\mathfrak{u}(g+1-n) \oplus \mathfrak{so}(2n))_c \oplus \left(\wedge^2 Q \oplus Q \otimes W\right) \oplus \overline{\left(\wedge^2 Q \oplus Q \otimes W\right)},$$

where $\wedge^2 Q \oplus Q \otimes W$ corresponds to the holomorphic tangent bundle $T^{1,0}\mathcal{F}_{g,n}$ of $\mathcal{F}_{g,n}$. Here $\wedge^2 Q$ is the holomorphic tangent bundle to the Hermitian fibres $SO(2g+2-2n)/U(g+1-n)$ of $\mathcal{F}_{g,n} \to \mathcal{G}_{g,n}$ and its complement $Q \otimes W$ is a holomorphic horizontal bundle (cf. [1]).

On the other hand,

$$T^{1,0}\mathcal{F}_{g,n}|_{\mathcal{M}_{g,n}} = T^{1,0}\mathcal{M}_{g,n} \oplus S^2 Q|_{\mathcal{M}_{g,n}},$$

so that

$$T = T^{1,0}\mathcal{M}_{g,n} = \wedge^2 Q \oplus Q \otimes W - S^2 Q,$$

where we are denoting bundles and their pull-backs by the same symbols.

Lemma 2.1.

$$T = Q \otimes W - \psi^2 Q,$$

where $\psi^2 = S^2 - \wedge^2$ *in* K*-theory.*

The operator ψ^2 is one of the series of Adams operators, defined by the formula

$$\sum_{p\geq 0}(\psi^p E)t^p = r - t\frac{\mathrm{d}}{\mathrm{d}t}\log\Lambda_{-t}E,$$

where $E \in K(\mathcal{M})$ has virtual rank r and $\Lambda_t E = \sum_{i\geq 0}(\wedge^i E)t^i$ [4]. Each ψ^p is a ring homomorphism in K-theory, and is characterized by the property that

$$\mathrm{ch}_k(\psi^p E) = p^k \mathrm{ch}_k(E), \tag{3}$$

where $\mathrm{ch}_k(E)$ denotes the term of dimension $2k$ in the Chern character. We can readily see that

$$c_1(T) = 2(n-1)c_1(L), \tag{4}$$

since $L = \det(Q)$ on $\mathcal{F}$. Note that L is a positive line bundle on $\mathcal{F}_{g,n}$ and, therefore, also on $\mathcal{M}_{g,n}$.

Since $Q^* + Q = 2g + 2 - W$ is a genuine complex vector bundle of rank $2(g+1-n)$ with total Chern class $c(W)^{-1}$, we obtain relations in the cohomology ring for dimension greater than $2(g+1-n)$.

3. Character calculations

Let $h = g + 1 - n$ and consider the holomorphic Euler characteristics

$$V_{h,n}(p,q,r) = \chi(\mathcal{M}_{g,n}, \mathcal{O}(\psi^{p-q}Q \otimes L^{q-(n-1)} \otimes \psi^r W)). \tag{5}$$

Let $w = x + x^{-1} - 2$ and

$$F(w,p) = \frac{(x^p - x^{-p})(x - x^{-1})}{x + x^{-1} - 2} = \sum_{h\geq 0}\left(4\binom{p+h}{2h+1} + \binom{p+h-1}{2h-1}\right)w^h. \tag{6}$$

Let $G(w,p)$ be such that $G(w,p)F(w,p) = 1$, i.e.

$$G(w,p) = \sum_{m=0}^{\infty}(-1)^m G_m(p)w^m = \sum_{m=0}^{\infty}\frac{(-1)^m P_m(p)}{(4p)^{m+1}w^m},$$

where

$$P_m(p) = \begin{cases} \sum_{j=1}^{2p-1}(-1)^{j+1}\left(\dfrac{p}{\sin^2(j\pi/2p)}\right)^m, & m \geq 0,\\ 0, & m < 0. \end{cases}$$

Theorem 3.1.

$$V_{h,n}(0,q,r) = 0, \qquad V_{h,n}(p,0,r) = 0,$$

$$V_{h,n}(p,q,r) = (-1)^h V_{h,n}(-p,-q,r), \qquad V_{h,n}(p,q,r) = -V_{h,n}(-p,q,r),$$

$$V_{h,n}(q,q,0)=\frac{2nh}{(4q)^{(h+1)(n-1)-g}}\begin{vmatrix} P_h(q) & P_{h+1}(q) & \cdots & P_{h+n-2}(q) \\ \vdots & \vdots & & \vdots \\ P_{h-n+2}(q) & P_{h-n+3}(q) & \cdots & P_h(q) \end{vmatrix}.$$

Remark.

$$\chi(\mathcal{M}_{g,n},\mathcal{O}(L^{q-(n-1)}))=\dim H^0(\mathcal{M}_{g,n},\mathcal{O}(L^{q-(n-1)}))$$

by Kodaira vanishing theorem since L is a positive line bundle on $\mathcal{M}_{g,n}$, Eq. (4) *and Serre duality. Thus,*

$$\dim H^0(\mathcal{M}_{g,n},\mathcal{O}(L^{q-(n-1)}))=\frac{V_{h,n}(q,q,0)}{2nh}$$

constitutes our Verlinde-type formula.

Remark. *Note that $V_{h,2}(q,q,0)/4h=P_h(q)$ is the original Verlinde formula*

$$P_h(q)=\dim(H^0(\mathcal{M}_{g,2},\mathcal{O}(L^{q-1}))),$$

$V_{h,n}(q,q,0)$ is the determinant of a matrix whose entries are Verlinde dimensions themselves, and the classic result

$$\frac{V_{h,1}(q,q,0)}{2h}=\dim(H^0(TJ(\Sigma),\mathcal{O}(L^q)))=(4q)^g.$$

Remark. *The second line in Theorem* 3.1 *represents the symmetries of $V_{h,n}(p,q,r)$.*

Proof. Let $\sigma^*=S^2Q$ and $m=\mathrm{rk}(S^2Q)$. We shall use the Koszul complex

$$\begin{aligned}&0\to\mathcal{O}_{\mathcal{F}}\left(\wedge^m\sigma\otimes V\right)\to\mathcal{O}_{\mathcal{F}}\left(\wedge^{m-1}\sigma\otimes V\right)\to\cdots\to\mathcal{O}_{\mathcal{F}}(\sigma\otimes V)\\ &\quad\to\mathcal{O}_{\mathcal{F}}(V)\to\mathcal{O}_{\mathcal{M}}(V)\to 0\end{aligned}\tag{7}$$

and the Atiyah–Bott fixed point theorem, where V is any virtual bundle over $\mathcal{F}_{g,n}$.

Recall that, on a homogeneous space, holomorphic Euler characteristics can be computed by means of the Atiyah–Bott fixed point formula. Let G be a reductive Lie group, P a parabolic subgroup of G, and $F=G/P$ the corresponding flag manifold. A representation R of P determines both a holomorphic vector bundle $\underline{R}=G\times_P R$ over F and a virtual G-module

$$\mathcal{I}_R=\sum_i(-1)^iH^i(F,\mathcal{O}(\underline{R})).$$

Let $\mathbb{T}$ be a common maximal torus of P and G, let W_G, and W_P be the Weyl groups, and W_{r} the relative Weyl group. The character of the G-module $\mathcal{I}_R$ is given by

$$\mathrm{tr}(\mathcal{I}_R)=\sum_{w\in W_{\mathrm{r}}}w\cdot\frac{\mathrm{tr}(R)}{\mathrm{tr}(\Lambda_{-1}A^*)},\tag{8}$$

where A is the P-module associated to the holomorphic tangent bundle $T^{1,0}\mathcal{F} = \underline{A}$. Evaluation at the identity element of $\mathbb{T}$ yields

$$\operatorname{tr}(\mathcal{I}_R)|_e = \chi(F, \mathcal{O}(\underline{R})).$$

Let B_1 and B_2 denote the fundamental representations of $U(h)$ and $SO(2n)$, respectively. The vector bundles Q, $\det(Q)$ and W over $\mathcal{F} = SO(2g+2)/(U(h)\times SO(2n))$ pull back to Q, L and W over $\mathcal{M}_{g,n}$, respectively. From the complex (7)

$$\chi(\mathcal{M}_{g,n}, \mathcal{O}(\psi^{p-q}Q \otimes \psi^r W \otimes L^{q-(n-1)})) = \chi(\mathcal{F}, \mathcal{O}(\underline{E}^{p,q,r})),$$

where

$$E^{p,q,r} = \psi^{p-q}B_1^* \otimes \psi^r B_2 \otimes (\det B_1^*)^{q-(n-1)} \otimes \Lambda_{-1}(S^2 B).$$

We shall now proceed to calculate (8). Let $x_1, \ldots, x_{g+1}$ be the characters of the maximal torus of $SO(2g+2)$ corresponding to the polarisation $\{y_{2j-1} + iy_{2j} : 1 \le j \le g+1\}$ of $\mathbb{C}^{2g+2}$. The character of the fundamental $SO(2g+2)$-module is $\sum_{j=1}^{h+n}(x_j + x_j^{-1})$, that of the fundamental $SO(2n)$-module is $\sum_{j=1}^{n}(x_{j+h} + x_{j+h}^{-1})$, and that of the fundamental $U(h)$-module is $\sum_{j=1}^{h} x_j^{-1}$. Thus,

$$\operatorname{tr}(E^{p,q,r}) = \prod_{i\le h} x_i^{q-(n-1)} \prod_{1\le j\le k\le h}\left(1 - \frac{1}{x_j x_k}\right)\left(\sum_{\ell=1}^{h} x_\ell^{p-q}\right)\left(\sum_{m=1}^{n} x_m^r + x_m^{-r}\right)$$

and

$$\operatorname{tr}(\Lambda_{-1}A^*) = \prod_{1\le i<j\le h}\left(1 - \frac{1}{x_i x_j}\right)\prod_{\substack{1\le k\le h\\ \varepsilon=1,\ldots,n}}\left(1 - \frac{1}{x_{h+\varepsilon}x_k}\right)\left(1 - \frac{x_{h+\varepsilon}}{x_k}\right).$$

Thus, we need to compute

$$\sum_{w\in W_{\mathrm{r}}} w \cdot \frac{\operatorname{tr}(E^{p,q,r})}{\operatorname{tr}(\Lambda_{-1}A^*)}. \tag{9}$$

We have

$$\frac{\operatorname{tr}(E^{p,q,r})}{\operatorname{tr}(\Lambda_{-1}A^*)} = \left(\prod_{i\le h} \frac{x_i^{q-(n-1)}(1 - x_i^{-2})}{\prod_{\varepsilon=1,\ldots,n}(1 - (1/x_{h+\varepsilon}x_k))(1 - (x_{h+\varepsilon}/x_k))}\right) \times \left(\sum_{j\le h} x_j^{p-q}\right)\left(\sum_{j=1}^{n} x_j^r + x_j^{-r}\right).$$

Using the identity

$$\left(1 - \frac{1}{yx}\right)\left(1 - \frac{y}{x}\right) = \frac{1}{x}\left(x + \frac{1}{x} - \left(y + \frac{1}{y}\right)\right),$$

$$\frac{\mathrm{tr}(E^{p,q,r})}{\mathrm{tr}(\Lambda_{-1}A^*)} = \left(\prod_{i\le h}\frac{x_i^q(x_i - x_i^{-1})}{\prod_{\varepsilon=1,\dots,n}(x_i + x_i^{-1} - x_{h+\varepsilon} - x_{h+\varepsilon}^{-1})}\right) \times \left(\sum_{j\le h} x_j^{p-q}\right)\left(\sum_{j=1}^{n} x_j^r + x_j^{-r}\right).$$

In order to perform the summation in (8), recall the form of the relative Weyl group W_{r} of $W_{SO(2g+2)}$ with respect to $W_{U(h)}$ and $W_{SO(2n)}$. Firstly,

$$W_{SO(2g+2)} = W_{2g+2}^{\mathrm{signs}} \rtimes W_{g+1}^{\mathrm{perms}},$$

where W^{signs} consists of substitutions $x_i \mapsto x_i^{-1}$ of an even number of variables from $\{x_1, \dots, x_{g+1}\}$, and W^{perms} is the group of permutations of the $g+1$ weights $x_1, \dots, x_{g+1}$. Secondly,

$$W_{SO(2n)} = W_{2n}^{\mathrm{signs}} \rtimes W_n^{\mathrm{perms}},$$

where W^{signs} consists of substitutions $x_i \mapsto x_i^{-1}$ of an even number of variables from $\{x_{g+2-n}, \dots, x_{g+1}\}$, and W^{perms} is the group of permutations of the n weights $x_{g+2-n}, \dots, x_{g+1}$. Thirdly,

$$W_{U(g+1-n)} = W_{g+1-n}^{\mathrm{perms}},$$

where W^{perms} is the group of permutations of the $(g+1-n)$ weights $x_1, \dots, x_{g+1-n}$.

Thus,

$$W_{\mathrm{r}} = W^{\mathrm{signs}} \rtimes W^{\mathrm{perms}}$$

and it has $2^h\binom{h+2}{2}$ elements, where W^{signs} consists of all the substitutions $x_i \mapsto x_i^{-1}$ of an even number of variables modulo $\{x_{h+1} \mapsto x_{h+1}^{-1}, \dots, x_{h+n} \mapsto x_{h+n}^{-1}\}$, and W^{perms} consists of all the cycles which permute elements of the two disjoint sets $\{x_1, \dots, x_{g+1-n}\}$ and $\{x_{g+2-n}, \dots, x_{g+1}\}$, and their products modulo the $W_{g+1-n}^{\mathrm{perms}}$ and W_n^{perms}.

Adding first with respect to W^{signs} we have

$$\left(\prod_{i\le h}\frac{(x_i - x_i^{-1})}{\prod_{1\le\varepsilon\le n}(x_i + x_i^{-1} - x_{h+\varepsilon} - x_{h+\varepsilon}^{-1})}\right)\left(\sum_{j\le h}(x_j^p - x_j^{-p})\prod_{l\ne j}(x_j^q - x_j^{-q})\right) \times \left(\sum_{m=1}^{n} x_{m+h}^r + x_{m+h}^{-r}\right) \qquad (10)$$

from which we immediately see that the holomorphic Euler characteristic vanishes if $p = 0$ or $q = 0$, yielding the first four identities of the theorem.

In order to prove the second part of the theorem, let $p = q,\ r = 0$. From (10) we have the expression

$$2nh\prod_{i\le h}\frac{(x_i - x_i^{-1})(x_i^q - x_i^{-q})}{\prod_{1\le\varepsilon\le n}(x_i + x_i^{-1} - x_{h+\varepsilon} - x_{h+\varepsilon}^{-1})}.$$

We can set one of the variables to 1, for example $x_{g+1} \to 1$, and adding with respect to W^{perms} gives

$$2nh\frac{\prod_{i=1}^{g}F(w_i,q)}{\mathrm{Vm}(w_1,\ldots,w_g)}\sum_{\sharp I=n-1}\frac{(-1)^{|I|+n(n-1)/2}\,\mathrm{Vm}(w_I)\mathrm{Vm}(w_{\hat{I}})}{\prod_{i_j\in I}F(w_{i_j},q)},$$

where $I = (i_1,\ldots,i_{n-1})$ is a multi-index with $1 \le i_1 < \cdots < i_{n-1} \le g$, $|I| = i_1 + \cdots + i_{n-1}$, $\hat{I}$ denotes its complement in $\{1,2,\ldots,q\}$, and Vm is the Vandermonde determinant in the given variables w_i. The last expression equals

$$2nh\prod_{i=1}^{g}F(w_i,q)\frac{\begin{vmatrix} G(w_1,q) & w_1G(w_1,q) & \cdots & w_1^{n-2}G(w_1,q) & 1 & w_1 & \cdots & w_1^{h-1}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ G(w_g,q) & w_gG(w_g,q) & \cdots & w_g^{n-2}G(w_g,q) & 1 & w_g & \cdots & w_g^{h-1}\end{vmatrix}}{\mathrm{Vm}(w_1,\ldots,w_g)},$$

whose limit when $\{w_i \to 0\}$ is

$$2nh(4q)^g\begin{vmatrix} G_h(q) & G_{h+1}(q) & \cdots & G_{h+n-2}(q)\\ \vdots & \vdots & & \vdots\\ G_{h-n+2}(q) & G_{h-n+3}(q) & \cdots & G_h(q)\end{vmatrix}.$$

□

Let

$$\frac{x}{\sinh(x)} = \sum_{i=0}^{\infty}C_{2i}\,x^{2i},$$

where

$$C_{2i} = \frac{1}{(2i)!}2^{2i}(2^{2i}-2)B_{2i}$$

and B_{2i} are the Bernoulli numbers. Recall that $P_h(q)$ is the coefficient of x^{3h} in

$$(-qx)^h\left(\frac{x}{\sinh(x)}\right)^{2h}\left(\frac{2qx}{\sinh(2qx)}\right)$$

(cf. [17]), so that the coefficient of q^{3h} in $P_h(q)$ is C_{2h}. The top power of q in $(1/2hn)\times V_{h,n}(q,q,0)$, $q^{(2n-1)(g+1-n)}$, has coefficient

$$\left\langle\frac{c_1(L)^{(2n-1)(g+1-n)}}{((2n-1)(g+1-n))!},[\mathcal{M}_{g,n}]\right\rangle,$$

which is given by a sum of products of all the leading coefficients of all the entries in the determinant in Theorem 3.1.

Proposition 3.1. *The symplectic volume of* $\mathcal{M}_{g,n}$ *for a symplectic form representing the cohomology class of* $c_1(L)$ *is*

$$\mathrm{v}(\mathcal{M}_{g,n}) = \frac{1}{4^{(h+1)(n-1)-g}} \begin{vmatrix} C_{2h} & C_{2h+2} & \cdots & C_{2h+2n-4} \\ \vdots & \vdots & & \vdots \\ C_{2h-2n+4} & C_{2h-2n+6} & \cdots & C_{2h} \end{vmatrix}.$$

4. Intersection numbers on $\mathcal{M}_{g,g-1}$

In this section, we shall restrict ourselves to the case $n = g-1$ for $g \geq 2$, in which the real Grassmannians $\mathcal{G}_g = \mathcal{G}_{g,g-1}$ are quaternionic Kähler manifolds [13,18].

4.1. Cohomology of the real Grassmannian

Let $\mathcal{G}_g = \mathcal{G}_{g,g-1}$ be the real Grassmannian ($g \geq 2$)

$$\mathcal{G}_g = \frac{SO(2g+2)}{SO(4)\,SO(2g-2)}$$

parametrising real oriented four-dimensional subspaces of $\mathbb{R}^{2g+2}$. The isotropy group is contained in $Sp(2g-2)Sp(1)$ making $\mathcal{G}_g$ into a quaternion-Kähler manifold [13,18]. Let $\hat{W}$ be the (complexified) tautological $SO(4)$-bundle over $\mathcal{G}_g$ and $\hat{W}^{\perp}$ be its orthogonal complement in the trivial bundle with fibre $\mathbb{R}^{2g+2}$. Note that $(\hat{W}^{\perp})_{\mathrm{c}}$ coincides with W of Section 2. The tangent bundle of $\mathcal{G}_g$ factors as

$$T\mathcal{G}_g = \hat{W} \otimes \hat{W}^{\perp}.$$

Since $SO(4) \cong Sp(1)Sp(1) \cong SU(2)SU(2)$,

$$\hat{W}_{\mathrm{c}} = U \otimes_{\mathrm{c}} V,$$

where $U,\ V$ are two copies of the fundamental representation of $SU(2)$, and the subscript c denotes complexification.

Thus,

$$(T\mathcal{G}_g)_{\mathrm{c}} = U \otimes (V \otimes W)$$

where U may be considered as a quaternionic line bundle and $V \otimes W$ as the complementary quaternionic bundle for $Sp(2g-2)$.

We shall consider the ring generated by the following classes

$$u = -c_2(U) \in H^4(\mathcal{G}_g), \qquad v = -c_2(V) \in H^4(\mathcal{G}_g).$$

Although u and v are not integral classes, their multiples $4u$, $4v$ are integral since the vector bundles S^2U, S^2V are globally defined, where S^2 denotes the second symmetric tensor power of the corresponding representation or bundle. Suppose that $4u = l^2$ and $4v = \hat{l}^2$, so that

$$\mathrm{ch}(U) = \mathrm{e}^{l/2} + \mathrm{e}^{-l/2} = 2 + u + \tfrac{1}{12}u^2 + \tfrac{1}{360}u^3 + \tfrac{1}{20160}u^4 + \cdots,$$
$$\mathrm{ch}(V) = \mathrm{e}^{\hat{l}/2} + \mathrm{e}^{-\hat{l}/2} = 2 + v + \tfrac{1}{12}v^2 + \tfrac{1}{360}v^3 + \tfrac{1}{20160}v^4 + \cdots$$

Later in the note, we shall need the following corollary to the Clebsh–Gordan formula, which is readily proved by induction.

Lemma 4.1. *Let $H \cong \mathbb{C}^2$ be the standard representation of $Sp(1) = SU(2)$. Let S^nH denote the nth symmetric tensor power of H. The tensor powers of the virtual representation $S^2H - 3$, where 3 denotes a trivial representation of dimension 3, satisfy*

$$(S^2H-3)^{\otimes m} = \sum_{j=0}^{m} \binom{2m+1}{j} S^{2(m-j)}H.$$

We know that $\mathcal{G}_g$ is a spin manifold [13], and therefore there is a Dirac operator D acting on sections of the spin bundle Δ. Let $E = V \otimes W$. Thus Δ decomposes as $\Delta_+ \oplus \Delta_-$, where

$$\Delta_+ = S^{2g-2}U \oplus S^{2g-4}U \otimes \wedge_0^2 E \oplus \cdots \oplus \wedge_0^{2g-2} E,$$
$$\Delta_- = S^{2g-3}U \otimes E \oplus S^{2g-5}U \otimes \wedge_0^3 E \oplus \cdots \oplus U \otimes \wedge_0^{2g-3} E$$

over $\mathcal{G}_g$. If F is a vector bundle over $\mathcal{G}_g$ equipped with a connection, one can extend the Dirac operator D to an elliptic operator with coefficients in F

$$D(F) : \Gamma(\Delta_+ \otimes F) \to \Gamma(\Delta_- \otimes F),$$

whose index is by definition $\mathrm{ind}(D(F)) = \dim(\ker D(F)) - \dim(\mathrm{coker}\, D(F))$.

Note that

$$\langle u^i v^j, [\mathcal{G}_g]\rangle = \langle u^j v^i, [\mathcal{G}_g]\rangle$$

due to the symmetry between the bundles U and V. We define the *quaternionic volume* of $\mathcal{G}_g$ to be

$$\mathrm{v}(\mathcal{G}_g) = \langle (4u)^{2g-2}, [\mathcal{G}_g]\rangle = \langle (4v)^{2g-2}, [\mathcal{G}_g]\rangle.$$

In order to compute the numbers $\langle u^i v^j, [\mathcal{G}_g]\rangle$, we need to compute the following indices.

Proposition 4.1. *Let $f_j(k) = \mathrm{ind}(D(S^{2g-2+2k}U \otimes S^{2j}V))$. Then*

$$f_j(k) = \frac{(2j+1)(2g+2k-1)(g+k+j)(g+k-1-j)}{g(g-1)(2g-1)(2g-2)} \times \binom{2g+k+j-2}{2g-3}\binom{2g+k-j-3}{2g-3}.$$

Proof. First, by twistor transform [10,13],

$$\mathrm{ind}(D(S^{2g-2+2k}U \otimes S^{2j}V)) = \chi(\mathcal{F}_g, \mathcal{O}(S^{2j}V \otimes L^k)),$$

where $\mathcal{F}_g = \mathcal{F}_{g,g-1}$. Thus, we apply the Borel–Weil–Bott theorem.

Let $V(\gamma)$ denote the complex irreducible representation of $SO(2g+2)$ with dominant weight γ, where $\gamma = (\lambda_1, \lambda_2, \ldots, \lambda_{g+1})$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{g+1} \geq 0$. For example, $V(1, 0, \ldots, 0) = \mathbb{C}^{2g+2}$ is the fundamental representation of $SO(2g+2)$ and $V(1, 1, 0, \ldots, 0) = \wedge^2\mathbb{C}^{2g+2} = \mathfrak{so}(2g+2, \mathbb{C})$ is the complexified adjoint representation.

Observe that Q corresponds to the fundamental representation $(1, 0)$ of $U(2)$, so that $L = \det(Q)$ and S^2Q correspond to the weights $(1, 1)$ and $(2, 0)$, respectively. Thus, $S^2V = S^2Q \otimes L^{-1}$ corresponds to $(1, -1)$. By embedding the maximal torus of $U(2)$ into the one of $SO(2g+2)$ in the obvious way, we see that the virtual representation

$$\sum_{r=0}^{4(g-1)+1} H^r(\mathcal{F}_g, \mathcal{O}(S^{2j}V \otimes L^k)) \cong V(k+j, k-j, 0, \ldots, 0),$$

and consequently,

$$\chi(\mathcal{F}_g, \mathcal{O}(S^{2j}V \otimes L^k)) = \dim V(k+j, k-j, 0, \ldots, 0).$$

The latter is easily computed by using the Weyl dimension formula

$$\dim V(\gamma) = \prod_{\alpha \in R_+} \frac{\langle \alpha, \delta + \gamma \rangle}{\langle \alpha, \delta \rangle},$$

where R_+ denotes the positive roots of $SO(2g+2)$:

$$R_+ = \{\mathrm{e}_i \pm \mathrm{e}_j,\ i < j\},$$

where $\{\mathrm{e}_i\}$ is the canonical basis of $\mathbb{R}^{2g+2}$, and $\delta = (g, g-1, g-2, \ldots, 1, 0)$. □

Proposition 4.2. *Evaluation on the fundamental class* $[\mathcal{G}_g]$ *yields*

$$\mathrm{v}(\mathcal{G}_g) = \frac{2}{g}\binom{4g-3}{2g-1},$$

and more generally,

$$\langle 4^{2g-2}u^{2g-2-j}v^j, [\mathcal{G}_g] \rangle = (-1)^j(4g-2j-3)\binom{2g-2}{j}\binom{4g-3}{2j+1}^{-1}\mathrm{v}(\mathcal{G}_g).$$

Proof. Observe that the index

$$\begin{aligned}\chi(\mathcal{F}_g, \mathcal{O}(L^k \otimes (S^2V-3)^{\otimes j})) &= \langle \mathrm{e}^{lk}(\mathrm{ch}(S^2V-3))^j \mathrm{td}(\mathcal{F}_g), [\mathcal{F}_g] \rangle \\ &= \sum_{i=0}^{j} (-1)^i \binom{2j+1}{i} f_{j-i}(k)\end{aligned}$$

by the Atiyah–Singer index theorem and Lemma 4.1. This is a polynomial in k for each j, which we denote by $h_j(k)$. Furthermore,

$$h_j(k) = (-1)^j \frac{(2j+1)!(2g-2j-4)!(2g+2k-1)}{j!g(2g-1)(2g-2-j)!} \times (2g^2+4gk-(j+2)g-2k+2k^2)\binom{2g+k-j-3}{2g-2j-4}\binom{2g-2+k}{2g-2}$$

and has degree $4g-3-2j$. Its leading term is, on the one hand,

$$\frac{2k^{4g-3-2j}}{(4g-3-2j)!}\langle 4^{2g-2}u^{2g-2-j}v^j, [\mathcal{G}_g]\rangle,$$

since the lowest-dimensional component of $\mathrm{ch}((S^2V-3)^{\otimes n})$ is v^n. On the other hand, the leading term is equal to

$$(-1)^j \frac{4(2j+1)!k^{4g-3-2j}}{j!g(2g-1)!(2g-2-j)!}. \qquad \square$$

Note that we have only missed the intersection numbers involving the extra cohomology class of $\mathcal{G}g$ which appears in dimension $2g-2$.

4.2. Intersection numbers on $\mathcal{M}_{g,g-1}$

The complex manifold $\mathcal{F}_g = \mathcal{F}_{g,g-1}$ has complex dimension $4g-3$, and parametrises complex two-dimensional subspaces Π of $\mathbb{C}^{2g+2}$ which are isotropic with respect to the standard $SO(2g+2)$-invariant bilinear form. It is a contact Kähler–Einstein manifold [10], which projects onto $\mathcal{G}_g$, $\pi : \mathcal{F}_g \to \mathcal{G}_g$, by sending Π to the four-dimensional subspace of $\mathbb{R}^{2g+2}$ whose complexification is $\Pi \oplus \overline{\Pi}$. Each fibre is isomorphic to a rational curve $SO(4)/U(2) \cong \mathbb{CP}^1$ in $\mathcal{F}_g$.

The Picard group $\mathrm{Pic}(\mathcal{F}_g)$ is generated by a line bundle $L \to \mathcal{F}_g$ such that [10]

1. $L|_{\pi^{-1}(x)} = \mathcal{O}(2)$ on $\pi^{-1}(x) \cong \mathbb{CP}^1$.
2. L^{2g-1} is isomorphic to the anticanonical bundle $K_{\mathcal{F}}^{-1}$ of $\mathcal{F}_g$.
3. If Q denotes the dual of the tautological $U(2)$-bundle over $\mathcal{F}_g$, $L = \det(Q)$.

Theorem 4.1. *The intersection numbers $\langle u^i v^j, [\mathcal{M}_{g,g-1}]\rangle$, where $i+j = 4g-6$, are skew-symmetric in u and v. Evaluating on the fundamental class $[\mathcal{M}_{g,g-1}]$ yields*

$$\langle u^{2g-3-j}v^j, [\mathcal{M}_{g,g-1}]\rangle = \frac{(-1)^j}{4^{2g-5}}\binom{2g-2}{j}\binom{4g-3}{2j+1}^{-1}\binom{4g-3}{2g-1}.$$

Proof. As a $(4g-6)$-dimensional submanifold of $\mathcal{F}_g$, $\mathcal{M}_{g,g-1}$ is Poincaré dual to the Euler class $c_3(S^2Q)$, which is easily computed from the identity $S^2Q = L \otimes \pi^* S^2 V$ and is equal to $4l(u-v)$. Hence,

$$\begin{aligned}\langle u^{2g-3-j}v^j,[\mathcal{M}_{g,g-1}]\rangle &= \langle 4lu^{2g-3-j}v^j(u-v),[\mathcal{F}_g]\rangle\\ &= 8\langle u^{2g-2-j}v^j-u^{2g-3-j}v^{j+1},[\mathcal{G}_g]\rangle\\ &= \frac{(-1)^j}{4^{2g-5}}\binom{2g-2}{j}\binom{4g-3}{2j+1}^{-1}\binom{4g-3}{2g-1},\end{aligned}$$

where the second equality follows from twistor transform. □

4.3. *Tangent relations and Newstead-type vanishings*

The holomorphic tangent bundle of $\mathcal{F}_g$ satisfies

$$T^{1,0}\mathcal{F}_g = Q\otimes W\oplus\wedge^2 Q = Q\otimes W\oplus L,$$

as in Section 2.

There is a local C^∞ isomorphism

$$\pi^*U = L^{1/2}\oplus L^{-1/2},$$

so that $l=c_1(L)\in H^2(\mathcal{F}_g,\mathbb{Z})$, and by the Leray–Hirsch theorem

$$\left(\frac{l}{2}\right)^2+\pi^*c_2(U)=0,$$

i.e. $l^2=4u$ (omitting π^*).

From Lemma 2.1,

$$T^{1,0}\mathcal{M}_{g,g-1}=Q\otimes W-\psi^2Q=(2g+2)\,V\otimes L^{1/2}-2\psi^2V\otimes L-2L-\psi^2V-2,$$

where we have omitted π^* and ψ^2 denotes the second Adams operator on vector bundles [4]. As in [6],

$$c(\mathcal{M}_{g,g-1})=\frac{((1+l/2+\hat{l}/2)(1+l/2-\hat{l}/2))^{2g+2}}{(1+l+\hat{l})^2(1+l-\hat{l})^2(1+l)^2(1-\hat{l}^2)},\tag{11}$$

where $\hat{l}$ is defined formally to be $2\sqrt{v}$ (we also denote by u and v the pull-backs to $\mathcal{M}_{g,g-1}$ of the quaternionic classes on $\mathcal{G}_g$). Thus,

$$p(\mathcal{M}_{g,g-1})=\frac{(1+2(u+v)+(u-v)^2)^{2g+2}}{(1+4u)^2(1+4v)^2(1+8(u+v)+16(u-v)^2)^2}\tag{12}$$

and

$$\begin{aligned}\hat{A}(\mathcal{M}_{g,g-1}) &= \left(\frac{\sqrt{u}+\sqrt{v}}{\sinh(\sqrt{u}+\sqrt{v})}\,\frac{\sqrt{u}-\sqrt{v}}{\sinh(\sqrt{u}-\sqrt{v})}\right)^{2g+2}\\ &\quad\times\left(\frac{\sinh(2(\sqrt{u}+\sqrt{v}))}{2(\sqrt{u}+\sqrt{v})}\,\frac{\sinh(2(\sqrt{u}-\sqrt{v}))}{2(\sqrt{u}-\sqrt{v})}\,\frac{\sinh(2\sqrt{u})}{2\sqrt{u}}\,\frac{\sinh(2\sqrt{v})}{2\sqrt{v}}\right)^2.\end{aligned}\tag{13}$$

The expressions (12) and (13) are symmetric in u and v. Hence, we have the following.

Corollary 4.1. *For $g \geq 2$, all the Pontrjagin numbers vanish as well as*

$$\hat{A}_{2g-3}(\mathcal{M}_{g,g-1}) = 0,$$

the Chern classes

$$c_{4g-6}(\mathcal{M}_{g,g-1}) = c_{4g-7}(\mathcal{M}_{g,g-1}) = 0$$

and in particular,

$$\chi(\mathcal{M}_{g,g-1}) = 0.$$

Furthermore,

$$\chi(\mathcal{M}_g, \mathcal{O}(T^{1,0}\mathcal{M}_g)) = \begin{cases} -1 & \text{if } g = 2, \\ -6 & \text{if } g = 3, \\ -2g+1 & \text{if } g \geq 4, \end{cases}$$

$$\chi(\mathcal{M}_g, \mathcal{O}(T^{0,1}\mathcal{M}_g)) = \begin{cases} -1 & \text{if } g \neq 3, \\ 2 & \text{if } g = 3. \end{cases}$$

Proof. The Chern class vanishings are proved by expanding the expression (11) and using the intersection numbers in Theorem 4.1.

The holomorphic Euler characteristics follow from the K-theoretical identity

$$T^{1,0}\mathcal{M}_{g,g-1} = (2g+2)V \otimes L^{1/2} - 2S^2V \otimes L - S^2V - 1$$

and the formulae of Proposition 4.1. Let (k) denote the operation of tensoring with L^k. Since $\mathcal{M}_{g,g-1}$ is the zero set of a non-degenerate section of the bundle $\sigma^* = S^2Q = S^2V(1)$, we have a Koszul complex

$$0 \to \mathcal{O}_{\mathcal{F}}(\wedge^3\sigma(k)) \to \mathcal{O}_{\mathcal{F}}(\wedge^2\sigma(k)) \to \mathcal{O}_{\mathcal{F}}(\sigma(k)) \to \mathcal{O}_{\mathcal{F}}(k) \to \mathcal{O}_{\mathcal{M}}(k) \to 0,$$

which is equivalent to

$$0 \to \mathcal{O}_{\mathcal{F}}(k-3) \to \mathcal{O}_{\mathcal{F}}(S^2V(k-2)) \to \mathcal{O}_{\mathcal{F}}(S^2V(k-1)) \to \mathcal{O}_{\mathcal{F}}(k) \to \mathcal{O}_{\mathcal{M}}(k) \to 0.$$

Tensoring the complex by V and S^2V, we see that

$$\chi(\mathcal{M}_{g,g-1}, \mathcal{O}(V(k))) = f_{1/2}(k) - f_{3/2}(k-1) - f_{1/2}(k-1) + f_{3/2}(k-2) + f_{1/2}(k-2) - f_{1/2}(k-3)$$

and

$$\chi(\mathcal{M}_{g,g-1}, \mathcal{O}(S^2V(k))) = f_1(k) - f_2(k-1) - f_1(k-1) - f_0(k-1) + f_2(k-2) + f_1(k-2) + f_0(k-2) - f_1(k-3),$$

respectively. □

Remark. *These vanishings constitute a generalisation of the Newstead conjectures to the spaces $\mathcal{M}_{g,g-1}$. In fact, the vanishings for $\mathcal{M}_{g,1}$ are due to the triviality of $TJ(\Sigma_g)$ and the* *vanishings for* $\mathcal{M}_{g,2}$ *were first found by Newstead* [11, Conjectures (a) and (b)] *and proved by Kirwan* [9] *and Gieseker* [5].

Conjecture 4.1. *The top* $(g+1-n)$ *Chern classes of* $\mathcal{M}_{g,n}$ *vanish, i.e.*

$$c_{(2n-2)(g+1-n)+j}(\mathcal{M}_{g,n}) = 0 \qquad \text{for} \quad j \geq 1.$$

References

[1] F.E. Burstall, J.R. Rawnsley, Twistor Theory for Riemannian Symmetric Spaces, Lecturer Notes in Mathematics, Vol. 1424, Springer, Berlin, 1990.

[2] S.K. Donaldson, Gluing techniques in the cohomology of moduli spaces, in: L.R. Goldberg, A.V. Philips (Eds.), Topological Methods in Modern Mathematics, Publish or Perish, Houston, 1993, pp. 137–170.

[3] G. Faltings, A proof for the Verlinde formula, J. Algebraic Geom. 3 (1994) 347–374.

[4] W. Fulton, S. Lang, Riemann–Roch Algebra, Springer, Berlin, 1985.

[5] D. Gieseker, A degeneration of the moduli space of stable bundles, J. Differential Geom. 19 (1984) 173–206.

[6] R. Herrera, Some Verlinde formulae and twistor transform, Preprint, 1999.

[7] R. Herrera, S. Salamon, Intersection numbers on moduli spaces and symmetries of a Verlinde formula, Commun. Math. Phys. 188 (3) (1997) 521–534, dg-ga/9612016.

[8] L.C. Jeffrey, F.C. Kirwan, Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface, Ann. Math. 148 (1) (1998) 109–196, alg-geom/9608029.

[9] F.C. Kirwan, The cohomology rings of moduli spaces of bundles over Riemann surfaces, J. Am. Math. Soc. 5 (1992) 853–906.

[10] C.R. LeBrun, S.M. Salamon, Strong rigidity of positive quaternion-Kähler manifolds, Invent. Math. 118 (1994) 109–132.

[11] P.E. Newstead, Characteristic classes of stable bundles over an algebraic curve, Trans. Am. Math. Soc. 169 (1972) 337–345.

[12] S. Ramanan, Orthogonal and Spin Bundles over Hyperelliptic Curves, Geometry and Analysis (papers dedicated to the memory of V.K. Patodi), Springer, Berlin, 1981.

[13] S.M. Salamon, Quaternionic Kähler manifolds, Invent. Math. 67 (1982) 143–171.

[14] S. Salamon, Harmonic and holomorphic maps, in: Seminar Luigi Bianchi II, Lecturer Notes in Mathematics, Vol. 1164, Springer, Berlin, 1990, pp. 161–224.

[15] S.M. Salamon, The twistor transform of a Verlinde formula, Riv. Mat. Univ. Parma 3 (1994) 143–157, dg–ga/9506003.

[16] A. Szenes, Hilbert polynomials of moduli spaces of rank 2 vector bundles I, Topology 32 (1993) 587–597.

[17] M. Thaddeus, Conformal field theory and the moduli space of stable bundles, J. Differential Geom. 35 (1992) 131–149.

[18] J.A. Wolf, Complex homogeneous contact structures and quaternionic symmetric spaces, J. Math. Mech. 14 (1965) 1033–1047.

[19] D. Zagier, On the cohomology of moduli spaces of rank two vector bundles over curves, in: R. Dijkgraaf, C. Faber, G. van der Geer (Eds.), The Moduli Spaces of Curves, Progress in Mathematics, Vol. 129, Birkhäuser, Basel, 1995, pp. 533–563.